

UNITARY REPRESENTATIONS OF RATIONAL CHEREDNIK ALGEBRAS, II

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ABSTRACT. The goal of this paper is to classify the unitary irreducible modules in category \mathcal{O}_c for the rational Cherednik algebras of type $G(r, 1, n)$. As a first step, we classify those irreducibles in \mathcal{O}_c that are diagonalizable with respect to a commutative subalgebra introduced by Dunkl and Opdam. As a byproduct, we obtain combinatorial character formulas for many irreducible modules in category \mathcal{O}_c , in terms of certain column-strict tableaux.

1. INTRODUCTION

1.1. The goal of this paper is to obtain the classification of irreducible unitary modules in category \mathcal{O}_c for the rational Cherednik algebras of type $G(r, 1, n)$. The strategy is that of the appendix to [EtSt]. The tactics are somewhat different, and the answer is quite a bit more complicated. The problem was posed by Cherednik, who also asked whether the socle of the polynomial representation is unitary for certain special parameters. The latter question was recently resolved by Feigin and Shramov in [FeSh].

The Cherednik algebra H_c contains a commutative subalgebra \mathfrak{t} that acts by locally finite, normal, and hence diagonalizable, operators on any unitary representation in category \mathcal{O}_c . Therefore a first step towards the classification of irreducible unitary modules in \mathcal{O}_c is the classification of the irreducible modules in \mathcal{O}_c on which \mathfrak{t} acts by diagonalizable operators (A. Ram [Ram] uses the word *calibrated* for the analog of this type of module for the affine Hecke algebra). Once we have achieved this classification (Theorem 2.1), the classification of unitary modules (Theorem 2.4) is obtained by Cherednik's technique of intertwining operators. For certain special examples associated to the groups $W = G(2, 1, n)$ we can be fairly explicit, it seems that a direct description of the unitary irreducibles in \mathcal{O}_c in general is unavoidably complicated.

In the appendix to [EtSt] we had the advantage that the first part (classification of diagonalizable modules) had been previously carried out by Cherednik and Suzuki for the trigonometric DAHA, and the corresponding classification for the rational DAHA relied on an embedding of the latter into the former. Unfortunately, we have no such embedding for complex reflection groups (nor even a definition of the trigonometric DAHA). So to obtain the classification of diagonalizable modules we work directly with the rational DAHA, making use of a presentation adapted to the technique of intertwining operators. Once this is done there are necessary and sufficient numerical criteria that the eigenvalues of a diagonalizable module must satisfy in order that it be unitary.

T. Suzuki remarks that in the type A case, the unitary modules correspond to integrable modules and the diagonalizable modules correspond to the *admissible* modules of Kac and Wakimoto via the Arakawa-Suzuki functor. We do not know the analogs of this coincidence for the groups $G(r, p, n)$, though it should be interesting to compare our results with those of Varagnolo and Vasserot, [VaVa].

We remark that the version of Clifford theory that appears in the last section of [Gri2] allows one to deduce analogous results for the groups $G(r, p, n)$ when $n > 2$. This paper therefore completes the classification of unitary modules in category \mathcal{O}_c for the Cherednik algebras attached to classical Weyl groups.

I am especially indebted to Arun Ram for teaching me the techniques employed in this paper, and to Ivan Cherednik, from whose papers I have learned a great deal. I thank Emanuel Stoica for useful suggestions which have improved the paper.

Finally, P. Etingof conjectures that the restrictions of unitary modules (resp., diagonalizable) modules remain unitary (resp., diagonalizable). Our results support this conjecture.

2. STATEMENT OF RESULTS

2.1. Let V be a complex vector space of dimension n and let $W \subseteq \mathrm{GL}(V)$ be a finite group of linear transformations of V . The set of *reflections* (sometimes called pseudo-reflections or complex reflections) in W is

$$T = \{s \in W \mid \dim(\mathrm{fix}(s)) = n - 1\}.$$

The group W is a *reflection group* if it is generated by T .

2.2. For each reflection $s \in T$ let c_s be a formal variable such that $c_{ws w^{-1}} = c_s$ for all $s \in T$ and $w \in W$, and choose $\alpha_s \in V^*$ such that the zero set of α_s is the fix space of s . Let $R = \mathbb{C}[c_s]_{s \in T}$ be the ring of polynomials generated by these variables (thus R is a polynomial ring in a set of variables corresponding to the conjugacy classes of reflections). Write $R[V] = R \otimes_{\mathbb{C}} \mathbb{C}[V]$ for the ring of polynomial functions on V with coefficients in R . For each $y \in V$ define a *Dunkl operator* on $R[V]$ by

$$(2.1) \quad y(f) = \partial_y(f) - \sum_{s \in T} c_s \langle \alpha_s, y \rangle \frac{f - s(f)}{\alpha_s} \quad \text{for } f \in R[V],$$

where ∂_y is the partial derivative of f in the direction y and $\langle \cdot, \cdot \rangle$ is the natural pairing between V^* and V . Each $g \in R[V]$ defines a multiplication operator $f \mapsto gf$ on $R[V]$, and the *rational Cherednik algebra* $H = H(W, V)$ determined by these data is the subalgebra of $\mathrm{End}_R(R[V])$ generated by W , $R[V]$, and the Dunkl operators $y \in V$.

It follows from, for example, [Gri1] Theorem 2.1 that these operators satisfy the relations

$$(2.2) \quad wyw^{-1} = w(y) \quad wxw^{-1} = w(x) \quad \text{for } w \in W, x \in V^*, \text{ and } y \in V,$$

$$(2.3) \quad yg - gy = \partial_y(g) - \sum_{s \in T} c_s \langle \alpha_s, y \rangle \frac{g - s(g)}{\alpha_s} s \quad \text{for } y \in V \text{ and } g \in R[V]$$

and

$$(2.4) \quad y_1 y_2 = y_2 y_1 \quad \text{for all } y_1, y_2 \in V.$$

In fact H is generated by $R[V]$, W , and $R[V^*]$ subject to (2.2) and the special case of (2.3) in which $g \in V^*$ is linear, in which case it may be written as

$$(2.5) \quad yx - xy = \langle x, y \rangle - \sum_{s \in T} c_s \langle \alpha_s, y \rangle \langle x, \alpha_s^\vee \rangle s$$

where $x - s(x) = \langle x, \alpha_s^\vee \rangle \alpha_s$ determines $\alpha_s^\vee \in V$.

The multiplication map induces an isomorphism

$$(2.6) \quad R[V] \otimes RW \otimes R[V^*] \rightarrow H$$

called the *triangular decomposition* of H .

2.3. Let $\{S^\lambda \mid \lambda \in \Lambda\}$ be the set of irreducible representations of $\mathbb{C}W$, where Λ is an index set. To avoid a profusion of subscripts, we abuse notation and write also S^λ for its extension to RW . Write $R[V^*] \rtimes W$ for the subalgebra of H generated by the Dunkl operators $y \in V$ and the group W . The *standard module* corresponding to $\lambda \in \Lambda$ is

$$(2.7) \quad \Delta(\lambda) = \text{Ind}_{R[V^*] \rtimes W}^H(S^\lambda)$$

where the $R[V^*] \rtimes W$ -module structure on S^λ is determined by $yS^\lambda = 0$ for all $y \in V$. Thanks to the triangular decomposition (2.6) there is an isomorphism of $R[V] \rtimes W$ -modules

$$(2.8) \quad R[V] \otimes_R S^\lambda \rightarrow \Delta(\lambda)$$

and via this isomorphism the Dunkl operators act according to the formula

$$(2.9) \quad y(f \otimes v) = \partial_y(f) \otimes v - \sum_{s \in T} c_s \langle \alpha_s, y \rangle \frac{f - s(f)}{\alpha_s} \otimes s(v) \quad \text{for } y \in V, f \in R[V], \text{ and } v \in S^\lambda.$$

2.4. There is a reparametrization that simplifies many calculations (in particular, the eigenvalues of the monodromy of the connections corresponding to the standard modules). Let \mathcal{A} be the set of hyperplanes H in V of the form $H = \text{fix}(s)$ for some $s \in T$. For each $H \in \mathcal{A}$ choose $\alpha_H \in V^*$ with H equal to the zero set of α_H . The subgroup $W_H = \{w \in W \mid w(v) = v \text{ if } v \in H\}$ is a cyclic subgroup, and we write W_H^\vee for its character group. Let $n_H = |W_H|$ be the size of W_H and for $\chi \in W_H^\vee$ let

$$e_{H,\chi} = \frac{1}{n_H} \sum_{w \in W_H} \chi(w^{-1})w \in \mathbb{C}W_H$$

be the corresponding primitive idempotent. Define $c_{H,\chi}$ by

$$c_{H,\chi} n_H = \sum_{s \in W_H - \{1\}} c_s (1 - \chi(s)).$$

In particular $c_{H,\text{triv}} = 0$, and R may be viewed as a polynomial ring in the variables $c_{H,\chi}$ (modulo the relations $c_{H,\chi} = c_{wH,w\chi w^{-1}}$ for $w \in W$). Using the relation

$$w = \sum_{\chi \in W_H^\vee} \chi(w) e_{H,\chi} \quad \text{for } w \in W_H,$$

the formula for Dunkl operators becomes

$$y(f) = \partial_y(f) - \sum_{H \in \mathcal{A}} \frac{\langle \alpha_H, y \rangle}{\alpha_H} \sum_{\chi \in W_H^\vee - \{1\}} c_{H,\chi} n_H e_{H,\chi},$$

which is, up to a sign, the formula in [DuOp], equation (5).

For each $H \in \mathcal{A}$ fix an eigenvector $\alpha_H^\vee \notin H$ for W_H . In terms of the parameters $c_{H,\chi}$ the relation (2.5) is

$$(2.10) \quad yx - xy = \langle x, y \rangle - \sum_{H \in \mathcal{A}} \frac{\langle \alpha_H, y \rangle \langle x, \alpha_H^\vee \rangle}{\langle \alpha_H, \alpha_H^\vee \rangle} \sum_{\chi \in W_H^\vee} (c_{H,\chi \otimes \det^{-1}} - c_{H,\chi}) n_H e_{H,\chi}$$

2.5. Fix a positive definite Hermitian inner product $\langle \cdot, \cdot \rangle$ on S^λ and mutually inverse W -equivariant conjugate linear isomorphisms $V \rightarrow V^*$ and $V^* \rightarrow V$, written $y \mapsto y^*$ and $x \mapsto x^*$. Then $\langle \cdot, \cdot \rangle$ has a unique extension, also denoted $\langle \cdot, \cdot \rangle$, to $\Delta(\lambda)$ determined by the following rules:

- (a) $\langle \cdot, \cdot \rangle$ is bi-additive, R -linear in the second variable, and R -conjugate linear in the first variable with respect to the extension of complex conjugation to R that fixes the variables $c_{H,\chi}$,
- (b) $\langle xf, g \rangle = \langle f, x^*g \rangle$ for all $x \in V^*$ and $f, g \in \Delta(\lambda)$.

2.6. We will consider two types of extensions of scalars: first, writing F for the fraction field of R , we write $H_F = F \otimes_R H$ for the *generic* Cherednik algebra, and similarly $\Delta_F(\lambda) = F \otimes_R \Delta(\lambda)$ and $\langle \cdot, \cdot \rangle_F$ for the F -conjugate linear extension of $\langle \cdot, \cdot \rangle$ to $\Delta_F(\lambda)$. Second, given a specialization $R \rightarrow \mathbb{C}$ of the variables c_s (or $c_{H,\chi}$) to complex numbers, we will write $H_c = \mathbb{C} \otimes_R H$ and $\Delta_c(\lambda) = \mathbb{C} \otimes_R \Delta(\lambda)$ for the corresponding specializations. We think of the symbol c as standing for this specialization, or equivalently, for a set of complex numbers indexed by conjugacy classes of reflections (or conjugacy classes of characters of rank one parabolic subgroups). In case the specialization is such that the variables $c_{H,\chi}$ are all real, the contravariant form also specializes and we write $\langle \cdot, \cdot \rangle_c$ for its specialization.

2.7. Suppose that we have specialized the variables to complex numbers c . *Category* \mathcal{O}_c is the subcategory of the category $H_c\text{-mod}$ of finitely generated H_c -modules on which each Dunkl operator $y \in V$ acts locally nilpotently. Thanks to (2.9) each standard module $\Delta_c(\lambda)$ is in \mathcal{O}_c . In fact, the quotient $L_c(\lambda)$ of the standard module $\Delta_c(\lambda)$ by its radical is simple and this gives a complete list of inequivalent irreducible objects in \mathcal{O}_c . Furthermore, the radical of $\Delta_c(\lambda)$ coincides with the radical of the form $\langle \cdot, \cdot \rangle_c$. Therefore the contravariant form descends to a non-degenerate form on $L_c(\lambda)$, and Cherednik has posed the problem of deciding when this form is positive definite.

2.8. From now on, we will assume that W is a monomial group. The precise definitions follow. Fix positive integers r and n . Let $W = G(r, 1, n)$ be the group of n by n matrices such that the entries are either 0 or a power of $e^{2\pi\sqrt{-1}/r}$, and there is exactly one non-zero entry in each row and each column. Then W is a group of matrices acting on $V = \mathbb{C}^n$, and we write y_1, \dots, y_n for the standard basis of V .

There are two W -orbits of reflecting hyperplanes: writing x_1, \dots, x_n for the standard basis of V^* , there those of the form $x_i = 0$ (for which $n_H = r$) and those of the form $x_i = \zeta^l x_j$ (for which $n_H = 2$). Write s_{ij} for the permutation matrix interchanging i and j and fixing all other coordinates, and ζ_i for the matrix that multiplies the i th coordinate by $\zeta = e^{2\pi\sqrt{-1}/r}$ and fixes all other coordinates. We will write $e_{ij} = \frac{1}{r} \sum_{l=0}^{r-1} \zeta^{-lj} \zeta_i^l$, and leave the other idempotents unnamed.

2.9. In the case $W = G(r, 1, n)$ the relations for the rational Cherednik algebra may be written in the following extremely explicit form. Let c_0 and $d_1, \dots, d_{r-1} \in \mathbb{C}$ be variables and let $R = \mathbb{C}[c_0, d_1, \dots, d_{r-1}]$. The Cherednik algebra H for W is generated by the algebras $R[y_1, \dots, y_n]$, $R[x_1, \dots, x_n]$, and RW , subject to the relations $wfw^{-1} = w(f)$ for $f \in R[y_1, \dots, y_n]$ or $f \in R[x_1, \dots, x_n]$ and $w \in W$,

$$(2.11) \quad y_i x_i = x_i y_i - c_0 \sum_{\substack{1 \leq j \neq i \leq n \\ 0 \leq l \leq r-1}} \zeta_i^l s_{ij} \zeta_i^{-l} - \sum_{l=0}^{r-1} (d_l - d_{l-1}) e_{il}$$

for $1 \leq i \leq n$, and

$$(2.12) \quad y_i x_j = x_j y_i + c_0 \sum_{l=0}^{r-1} \zeta^{-l} \zeta_i^l s_{ij} \zeta_i^{-l}$$

for $1 \leq i \neq j \leq n$. To define d_l for all $l \in \mathbb{Z}$ we specify d_0 by the relation $d_0 + d_1 + \dots + d_{r-1} = 0$ and impose $d_i = d_j$ if $i = j \pmod r$. This fixes the precise relation between these parameters and the preceding $c_{H,\chi}$ (though we do not write it explicitly here). Whenever we make use of the contravariant form, we will assume the parameters c_0 and d_l are all real.

2.10. A *partition* $\lambda = \lambda_1 \geq \lambda_2 \geq \dots$ is a weakly decreasing sequence of integers such that $\lambda_n = 0$ for n large enough. Given a positive integer r , an r -partition is a sequence $\lambda^\bullet = (\lambda^0, \lambda^1, \dots, \lambda^{r-1})$ of r partitions. The *size* of an r -partition λ^\bullet is the sum $|\lambda^\bullet| = \sum_{i,j} \lambda_j^i$, and an r -partition of n is an r -partition λ^\bullet with $|\lambda^\bullet| = n$. We picture partitions and r -partitions as *Young diagrams*: collections

of boxes stacked in a corner, as in (2.13) (but without the numbers). A *tableau* on an r -partition λ^\bullet is a function T from the boxes of λ^\bullet to the integers. A *standard tableau* on an r -partition λ^\bullet of n is a bijection from the boxes of λ^\bullet to $\{1, 2, \dots, n\}$ such that the entries in each λ^i are strictly increasing left to right and top to bottom. An example of a standard tableau on the 2-partition $\lambda^\bullet = ((3, 2), (2, 2))$ of 9 is

$$(2.13) \quad \left(\begin{array}{|c|c|c|} \hline 2 & 4 & 6 \\ \hline 3 & 9 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 7 & 8 \\ \hline \end{array} \right).$$

Given a box $b \in \lambda^\bullet$, define $\beta(b) = l$ if $b \in \lambda^l$ and $\text{ct}(b) = j - i$ if b is in the i th row and j th column of λ^l . For the example (2.13) we have $\beta(T^{-1}(5)) = 1$ and $\text{ct}(T^{-1}(5)) = 1$.

2.11. Let $W = G(r, 1, n)$. The *Jucys-Murphy elements* of the group algebra $\mathbb{C}W$ are

$$(2.14) \quad \phi_i = \sum_{\substack{1 \leq j < i \\ 0 \leq l \leq r-1}} \zeta_i^l s_{ij} \zeta_i^{-l} \quad \text{for } 1 \leq i \leq n.$$

Together with the elements ζ_i of W they generate a subalgebra of $\mathbb{C}W$ that acts diagonalizably on every W -module. There is a bijection $\lambda^\bullet \mapsto S^{\lambda^\bullet}$ from the set of r -partitions of n to the set of irreducible W -modules such that S^{λ^\bullet} has a basis v_T indexed by standard Young tableaux T on λ^\bullet , and v_T is determined up to scalars by the equations

$$(2.15) \quad \phi_i v_T = \text{rct}(T^{-1}(i)) v_T \quad \text{and} \quad \zeta_i v_T = \zeta^{\beta(T^{-1}(i))} v_T \quad \text{for } 1 \leq i \leq n.$$

We fix a W -invariant positive definite Hermitian form on each S^{λ^\bullet} and assume that the norm of v_T with respect to this form is 1. For the groups $G(2, 1, n)$, it seems these versions of Jucys-Murphy elements were first written down in [Che2].

2.12. As in [DuOp] Definition 3.7, we put

$$(2.16) \quad z_i = y_i x_i + c_0 \phi_i \quad \text{for } 1 \leq i \leq n.$$

Together with the elements ζ_i they generate a commutative algebra \mathfrak{t} of H_c , and Theorem 5.1 of [Gri2] states that \mathfrak{t} acts on each standard module $\Delta_c(\lambda^\bullet)$ in an upper-triangular fashion.

2.13. We define a partial order on the boxes of λ^\bullet by: $b \leq b'$ if $T(b) < T(b')$ for all standard Young tableaux T on λ^\bullet . Thus $b \leq b'$ if and only if $\beta(b) = \beta(b')$ and b is (weakly) up and to the left of b' .

We write $\Gamma = \Gamma(\lambda^\bullet)$ for the set of pairs (P, Q) of tableaux on λ^\bullet such that P is a bijection from the boxes of λ^\bullet to the set $\{1, 2, \dots, n\}$, Q is a filling of the boxes of λ^\bullet by non-negative integers such that if $b < b'$ then $Q(b) \leq Q(b')$, with $Q(b) = Q(b')$ implying $P(b) > P(b')$. Then Theorem 5.1 of [Gri2] implies that there is a \mathfrak{t} -eigenbasis $f_{P,Q}$ of $\Delta(\lambda^\bullet)$ such that

$$\zeta_i f_{P,Q} = \zeta^{\beta(P^{-1}(i))} f_{P,Q}$$

and

$$z_i f_{P,Q} = (Q(P^{-1}(i)) + 1 - (d_{\beta(P^{-1}(i))} - d_{\beta(P^{-1}(i)) - Q(P^{-1}(i)) - 1}) - \text{rct}(P^{-1}(i)) c_0) f_{P,Q}.$$

The indexing here is related to that in the paper [Gri2] as follows: for a pair (μ, T) consisting of a standard Young tableau T on λ^\bullet and $\mu \in \mathbb{Z}_{\geq 0}^n$, we let w_μ be the longest element of S_n such that $w_\mu(\mu)$ is non-decreasing and define $P = w_\mu^{-1} T$ and $Q(b) = \mu_{P(b)}$. Note that we used an unorthodox convention for standard Young tableaux in [Gri2], regarding them as functions from $\{1, 2, \dots, n\}$ to the boxes of λ^\bullet (we have now come to our senses). This gives a bijection from the set of pairs (μ, T) as above to Γ .

2.14. Our first main result is the classification and description of the modules $L_c(\lambda^\bullet)$ that are \mathfrak{t} -diagonalizable. For each box $b \in \lambda$, define statistics $c(b)$, $k_c(b)$ and $l_c(b)$ as follows:

$$(2.17) \quad c(b) = d_{\beta(b)} + \text{rct}(b)c_0,$$

$k_c(b)$ is the smallest positive integer k such that there is a box $b' \in \lambda^{\beta(b)-k}$ with

$$k = c(b) - c(b'),$$

(and $k_c(b) = \infty$ if no such equation holds) and $l_c(b)$ is the smallest positive integer l such that there is an outside addable box b' for $\lambda^{\beta(b)-l}$ with

$$l = c(b) - c(b')$$

(and $l_c(b) = \infty$ if no such equation holds). Recall that a box b is *addable* to a partition λ if $b \notin \lambda$ and adding b to λ produces the diagram of a partition. An *outside* addable box is an addable box b such that $\text{ct}(b) \neq \text{ct}(b')$ for all $b' \in \lambda$. The statistic $c(b)$ may be thought of as a sort of “charged content” of the box b .

We define a subset $\Gamma_c \subseteq \Gamma$ as follows: a pair $(P, Q) \in \Gamma$ is in Γ_c if and only if the following conditions hold:

- (a) whenever $b \in \lambda^\bullet$ and $k \in \mathbb{Z}_{>0}$ with $k = d_{\beta(b)} - d_{\beta(b)-k} + \text{rct}(b)c_0$ we have $Q(b) < k$, and
- (b) whenever $b_1, b_2 \in \lambda^\bullet$ and $k \in \mathbb{Z}_{>0}$ with $\beta(b_1) - \beta(b_2) = k \bmod r$ and $k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\text{ct}(b_1) - \text{ct}(b_2) \pm 1)c_0$ we have $Q(b_1) \leq Q(b_2) + k$, with equality implying $P(b_1) > P(b_2)$.

Theorem 2.1. *The module $L_c(\lambda^\bullet)$ is diagonalizable if and only if either*

- (a) $c_0 = 0$ or
- (b) $c_0 \neq 0$ and for every removable box $b \in \lambda^\bullet$, either $k_c(b) = \infty$ or the inequality $l_c(b) < k_c(b)$ holds.

In case (b), as basis of $L_c(\lambda^\bullet)$ is given by the set $\{f_{(P,Q)} \mid (P, Q) \in \Gamma_c\}$ of non-symmetric Specht-valued Jack polynomials.

The modules $\Delta_c(\lambda^\bullet)$ are graded by polynomial degree, and the modules $L_c(\lambda^\bullet)$ are graded quotients (thanks to the deformed Euler operator from [DuOp]), on which W acts preserving the degree. Writing $L_c^d(\lambda^\bullet)$ for the degree d piece of $L_c(\lambda^\bullet)$, we define the graded character

$$\text{char}(L_c(\lambda^\bullet)^W, q) = \sum_{d=0}^{\infty} \dim_{\mathbb{C}}(L_c^d(\lambda^\bullet)^W) q^d.$$

By using Theorem 2.1 of [DuGr], we obtain the following corollary, where we define a column-strict tableau on λ^\bullet to be a filling of its boxes by non-negative integers in such a way that within each component λ^i , the entries are weakly increasing left to right, and strictly increasing top to bottom.

Corollary 2.2. *If $c_0 \neq 0$ and $L_c(\lambda^\bullet)$ is diagonalizable, then the graded character of $L_c(\lambda^\bullet)^W$ is given by the formula*

$$\text{char}(L_c(\lambda^\bullet)^W, q) = \sum_{d=0}^{\infty} a_d q^d,$$

where a_d is the number of column-strict tableaux Q on λ^\bullet such that $Q(b) = \beta(b) \bmod r$ for all boxes b of λ , the sum

$$\sum_{b \in \lambda^\bullet} Q(b) = d,$$

and for each positive integer l the following conditions hold:

- (a) *we have $Q(b) < l$ whenever $b \in \lambda^i$ and the equation $d_i - d_{i-l} + \text{rct}(b)c_0 = l$ holds, and*

(b) we have $Q(b_1) - Q(b_2) \leq l$ whenever $b_1 \in \lambda^i$, $b_2 \in \lambda^{i-l}$, and one of the equations

$$d_i - d_{i-l} + r(\text{ct}(b_1) - \text{ct}(b_2) \pm 1)c_0 = l$$

holds.

Writing $e = \frac{1}{|W|} \sum_{w \in W} w$ for the symmetrizing idempotent, if the parameter c is not on one of the hyperplanes specified by Theorem 3.4 of [DuGr], then the functor $M \mapsto eM = M^W$ from H_c modules to $eH_c e$ -modules is an equivalence, and the module $L_c(\lambda^\bullet)$ is finite dimensional if and only if $L_c(\lambda^\bullet)^W$ is.

2.15. Our second main result is the classification of the modules $L_c(\lambda^\bullet)$ that are unitary. First we deal with the case $c_0 = 0$. Let m_{ij} be the integer determined by $m_{ij} = i - j \bmod r$ and $1 \leq m_{ij} \leq r$. The following proposition is an immediate consequence of the norm formula for the Specht-valued Jack polynomials, Theorem 6.1 of [Gri2].

Proposition 2.3. *If $c_0 = 0$ then $L_c(\lambda^\bullet)$ is unitary if and only if for each i such that $\lambda^i \neq \emptyset$, the following condition holds: for each $0 \leq j \leq r-1$ with $d_i - d_j > m_{ij}$ there is some k with $m_{ik} < m_{ij}$ and $d_i - d_k = m_{ik}$.*

Let $c = (c_0, d_1, \dots, d_r)$ be a parameter, and let $c' = (-c_0, d_1, \dots, d_r)$. The maps

$$x \mapsto x, \ y \mapsto y, \ s_{ij} \mapsto -s_{ij}, \ \text{and} \ \zeta_l \mapsto \zeta_l \quad \text{for } x \in V^*, \ y \in V, \ 1 \leq i < j \leq n \text{ and } 1 \leq l \leq n$$

extend to an isomorphism of H_c onto $H_{c'}$ by which the module $L_c(\lambda^0, \lambda^1, \dots, \lambda^{r-1})$ corresponds to $L_{c'}((\lambda^0)', (\lambda^1)', \dots, (\lambda^{r-1})')$, where λ' is the transpose of λ . Therefore we may assume $c_0 > 0$ without loss of generality.

Let $b \in \lambda^i$ and let j be an integer with $0 \leq j \leq r-1$. A *preblocking sequence* for (b, j) is a sequence $B = (b_0, b_1, \dots, b_{2q+1}, l)$ of boxes $b_k \in \lambda^\bullet$ and an integer $0 \leq l \leq r-1$ such that

- (a) $b_0 = b$,
- (b) for $0 \leq k \leq q$, $b_{2k} \leq b_{2k+1}$, and
- (c) for $1 \leq k \leq q$, the numbers $m_{i, \beta(b_{2k})}$ are strictly increasing, and $m_{i, \beta(b_{2q})} < m_{il} \leq m_{ij}$ unless $q = 0$.

If we picture the r -partition λ^\bullet as a cycle or necklace of partitions, with λ^i positioned at the top, and with the partitions $\lambda^{i-1}, \lambda^{i-2}, \dots, \lambda^{i+1}$ appearing in sequence in the counterclockwise direction from λ^i , then condition (c) is equivalent to the boxes b_{2k} for $k = 1, 2, \dots, q$ appearing in counterclockwise sequence strictly between λ^i and λ^j , and l appearing strictly after the last of these and weakly before λ^j . It is also equivalent to the inequality

$$m_{\beta(b_{2q+1}), l} + \sum_{k=1}^q m_{\beta(b_{2k-1}), \beta(b_{2k})} \leq m_{ij}.$$

A preblocking sequence for (b, j) is a *blocking sequence* if

$$d_{\beta(b_{2q+1})} - d_l + r \text{ct}(b_{2q+1})c_0 = m_{\beta(b_{2q+1}), l}$$

and for each $1 \leq k \leq q$,

$$d_{\beta(b_{2k-1})} - d_{\beta(b_{2k})} + r(\text{ct}(b_{2k-1}) - \text{ct}(b_{2k}) \pm 1)c_0 = m_{\beta(b_{2k-1}), \beta(b_{2k})}.$$

A *preblocking sequence* for a pair (b, b') of boxes of λ^\bullet is either a preblocking sequence for b and $\beta(b')$ (as defined above), or a sequence $B = (b_0, b_1, \dots, b_{2q+1})$ satisfying (a) and (b) above, with (c) replaced by

- (d) $b_{2q+1} = b'$, and for $1 \leq k \leq q$ the numbers $m_{i, \beta(b_{2k})}$ are strictly increasing.

Picturing λ^\bullet as a cycle of partitions as above, this condition means the boxes b_{2k} appear in strict counterclockwise order between λ^i and λ^j .

A preblocking sequence for (b, b') is a *blocking sequence* if it is one for $(b, \beta(b'))$, or it is of the form $B = (b_0, \dots, b_{2q+1})$ and for each $1 \leq k \leq q$

$$d_{\beta(b_{2k-1})} - d_{\beta(b_{2k})} + r(\text{ct}(b_{2k-1}) - \text{ct}(b_{2k}) \pm 1)c_0 = m_{\beta(b_{2k-1}), \beta(b_{2k})}.$$

Theorem 2.4. *Suppose $c_0 > 0$. The module $L_c(\lambda^\bullet)$ is unitary if and only if the following conditions are satisfied:*

- (a) *it is diagonalizable (see Theorem 2.1),*
- (b) *for every pair $b, b' \in \lambda^\bullet$ of boxes such that*

$$d_{\beta(b)} - d_{\beta(b')} + r(\text{ct}(b) - \text{ct}(b') + 1)c_0 > m_{\beta(b), \beta(b')} > d_{\beta(b)} - d_{\beta(b')} + r(\text{ct}(b) - \text{ct}(b') - 1)c_0$$

there is a blocking sequence, and

- (c) *for every box $b \in \lambda^\bullet$ and $0 \leq j \leq r-1$ such that*

$$d_{\beta(b)} - d_j + r\text{ct}(b)c_0 > m_{\beta(b), j}$$

there is a blocking sequence.

The theorem exhibits the set of parameters c for which the module $L_c(\lambda^\bullet)$ is unitary as a (in practice, quite complicated) semi-linear subset of the parameter space. We are somewhat disappointed with the answer: in examples, and in fact already for $r = 2$ and $\lambda = (\lambda^0, \lambda^1)$ with both components non-empty, it is quite tedious to draw the unitary set. It is a closed semi-linear subset of the plane, but we have not found an expression for it much simpler than the one given by the previous theorem.

2.16. For $r = 1$ the theorems have as corollaries the results of Cherednik [Che1] and Suzuki [Suz] classifying the diagonalizable irreducible modules, and one of the results of [EtSt], classifying unitary modules for the Cherednik algebra of the symmetric group. In this case, the parameter is a number c , and given a partition λ we let b_1 be the removable box of λ with largest content a . Then, if $b = c_{\min}$ is the smallest number which is the content of a box of λ , it is a consequence of Theorem 2.1 that for $c > 0$, the module $L_c(\lambda)$ is diagonalizable exactly if c is not a positive rational number with denominator at most $a - b$: if so, $k_c(b) = l_c(b) < \infty$, and if not $k_c(b) = \infty$. For $c < 0$ replace λ by its transpose.

2.17. For $r = 2$ our results can be made simpler in some special cases: we will assume $\lambda^\bullet = (\lambda, \emptyset)$ where λ is a partition of n . The results that follow are not difficult to deduce from Theorems 2.1 and 2.4.

Corollary 2.5. *Assume $c_0 > 0$. If $\lambda^\bullet = (\lambda, \emptyset)$, we write c_{\min} for the minimal content of a box of λ , and we write b_1 for the removable box of λ of maximal content c_1 , then $L_c(\lambda^\bullet)$ is diagonalizable if and only if*

- (a) *c_0 is not a rational number of denominator at most $c_1 - c_{\min}$, or*
- (b) *$c_0 = k/l$ for coprime positive integers k and l such that $l \leq c_1 - c_{\min}$, there is no removable box b_2 in λ of content c_2 satisfying $l \leq c_2 - c_{\min}$, and an equation of the form*

$$d_0 - d_1 + 2\text{ct}(b_1)c_0 = m$$

holds for some positive odd integer $m < 2k$.

For the statement of the next corollary we make further non-degeneracy assumptions about the shape of λ . We omit the cases these assumptions rule out only for the sake of brevity; the results of are the same general type.

Corollary 2.6. *With the notation of the preceding corollary, assume that λ has removable boxes b_1 and b_2 such that $\text{ct}(b_1) > \text{ct}(b_2) > c_{\min}$, and with no other removable box of content larger than $\text{ct}(b_2)$. The set of parameters (c_0, d_0) such that $c_0 > 0$ and $L_c(\lambda, \emptyset)$ is diagonalizable is the union of the following sets:*

(a) *The set of parameters (c_0, d_0) satisfying the inequalities*

$$c_0 < \frac{1}{c_{\max} - c_{\min} + 1} \quad \text{and} \quad d_0 - d_1 + 2c_{\max}c_0 < 1,$$

(b) *for each positive integer l with $\text{ct}(b_1) - c_{\min} < l \leq c_{\max} - c_{\min} + 1$, the points (c_0, d_0) such that $c_0 = 1/l$ and*

$$d_0 - d_1 + 2c_{\max}c_0 < 1,$$

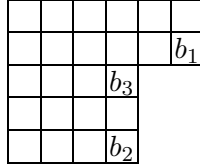
(c) *for each positive integer l with $\text{ct}(b_1) < l \leq c_{\max}$, the points (c_0, d_0) with*

$$d_0 - d_1 + 2lc_0 = 1 \quad \text{and} \quad c_0 < \frac{1}{l - c_{\min}},$$

(d) *the points (c_0, d_0) such that $d_0 - d_1 + 2\text{ct}(b_1)c_0 = 1$ and such that either*

$$c_0 \leq \frac{1}{\text{ct}(b_3) - c_{\min} + 1} \quad \text{or} \quad c_0 = \frac{1}{l - c_{\min} + 1}$$

for an integer l satisfying $\text{ct}(b_2) \leq l \leq \text{ct}(b_3)$, where b_3 is the box in the right rim of λ with maximal content subject to $\text{ct}(b_3) < \text{ct}(b_1)$. See the figure below.



3. THE TRIGONOMETRIC PRESENTATION OF H

Here we give another presentation of H which is adapted to the application of intertwining operators to the classification of diagonalizable modules in the next section.

3.1. The affine Weyl semigroup. Let $W_{\geq 0} = \mathbb{Z}_{\geq 0}^n \rtimes S_n$. It contains the elements s_1, \dots, s_{n-1} and $\Phi = \epsilon_n s_{n-1} \cdots s_2 s_1$, so that s_1, \dots, s_{n-1} satisfy the usual Coxeter relations, and interact with Φ via the relations

$$(3.1) \quad \Phi s_i = s_{i-1} \Phi \text{ for } 2 \leq i \leq n-1 \text{ and } \Phi^2 s_1 = s_{n-1} \Phi^2.$$

In fact, the abstract semigroup with generators s_1, \dots, s_{n-1} and Φ , together with the Coxeter relations and (3.1) is isomorphic to $W_{\geq 0}$, as we now sketch.

Letting G be this semigroup, it follows that there is a map $G \rightarrow W_{\geq 0}$ and thus that s_1, \dots, s_{n-1} generate a copy of S_n inside G . Define $\epsilon_n = \Phi s_1 \cdots s_{n-1}$. The relations in (3.1) imply that $s_i \epsilon_n = \epsilon_n s_i$ for $1 \leq i \leq n-2$, and therefore we may unambiguously define $\epsilon_i = w \epsilon_n w^{-1}$ for each $1 \leq i \leq n-1$ and any $w \in S_n$ with $w(n) = i$. It follows from this definition that $w \epsilon_i w^{-1} = \epsilon_{w(i)}$ for all $1 \leq i \leq n$ and $w \in S_n$. Again using (3.1), a direct calculation shows that $\epsilon_n \epsilon_1 = \epsilon_1 \epsilon_n$, and hence for all $1 \leq i \neq j \leq n$ choosing $w \in S_n$ with $w(1) = i$ and $w(n) = j$ gives $\epsilon_i \epsilon_j = w \epsilon_1 \epsilon_n w^{-1} = w \epsilon_n \epsilon_1 w^{-1} = \epsilon_j \epsilon_i$. It follows from this that there is a map $W_{\geq 0} \rightarrow G$ inverse to the previous one.

3.2. The Dunkl-Opdam subalgebra. The *Dunkl-Opdam* subalgebra \mathfrak{t} of H is the (commutative, as proved in [DuOp]) subalgebra of H generated by z_1, \dots, z_n and ζ_1, \dots, ζ_n . By the PBW theorem it is isomorphic to the polynomial ring in the variables z_1, \dots, z_n tensored with the group algebra of $(\mathbb{Z}/r\mathbb{Z})^n$. Define an automorphism ϕ of \mathfrak{t} by

$$(3.2) \quad \phi(\zeta_i) = \zeta_{i+1} \quad \text{for } 1 \leq i \leq n-1 \quad \text{and} \quad \phi(\zeta_n) = \zeta^{-1}\zeta_1$$

and

$$(3.3) \quad \phi(z_i) = z_{i+1} \quad \text{for } 1 \leq i \leq n-1 \quad \text{and} \quad \phi(z_n) = z_1 + 1 - \sum_{j=0}^{r-1} (d_{j-1} - d_{j-2})e_{1j}.$$

Put $\Phi = x_n s_{n-1} \cdots s_1$ and $\Psi = y_1 s_1 \cdots s_{n-1}$. By Proposition 4.3 and Lemma 5.3 of [Gri2],

$$(3.4) \quad f\Phi = \Phi\phi(f) \quad \text{and} \quad f\Psi = \Psi\phi^{-1}(f)$$

for all $f \in \mathfrak{t}$, and

$$(3.5) \quad z_i s_j = s_j z_i \quad \text{if } j \neq i, i+1, \text{ while } \quad z_i s_i = s_i z_{i+1} - c_0 \pi_i \quad \text{for } 1 \leq i \leq n-1,$$

where

$$\pi_i = \sum_{l=0}^{r-1} \zeta_i^l \zeta_{i+1}^{-l}.$$

Thus (as observed in [Dez1] and [Dez2]) the subalgebra H_{gr} of H generated by \mathfrak{t} and $W = G(r, 1, n)$ is isomorphic to the generalized graded affine Hecke algebra for $G(r, 1, n)$ defined in [RaSh]. The structure of an H_{gr} -module may be put on a vector space by defining an action of \mathfrak{t} together with operators s_i satisfying the Coxeter relations together with (3.5) and $w\zeta_i = \zeta_{w(i)}w$ for all $w \in S_n$ and $1 \leq i \leq n$.

3.3. The trigonometric presentation. H contains the commutative subalgebra \mathfrak{t} , elements Φ , Ψ , and s_1, \dots, s_{n-1} . These satisfy the following relations: (1) \mathfrak{t} and s_1, \dots, s_{n-1} generate a graded affine Hecke algebra H_{gr} for the group $G(r, 1, n)$ inside H , (2) Φ and s_1, \dots, s_{n-1} generate an affine Weyl semigroup inside H , (3) Ψ and s_1, \dots, s_{n-1} generate an affine Weyl semigroup inside H with relations

$$(3.6) \quad \Psi s_i = s_{i+1} \Psi \quad \text{for } 1 \leq i \leq n-2 \quad \text{and} \quad \Psi^2 s_{n-1} = s_1 \Psi^2,$$

and the following relations hold:

$$(3.7) \quad \zeta_i \Phi = \Phi \phi(\zeta_i), \quad \zeta_i \Psi = \Psi \phi^{-1}(\zeta_i),$$

$$(3.8) \quad \Psi \Phi = z_1, \quad \Phi \Psi = z_n - \kappa + \sum_{j=0}^{r-1} (d_j - d_{j-1})e_{nj}, \quad \text{and} \quad \Psi s_{n-1} \Phi = \Phi s_1 \Psi + c_0 \sum_{0 \leq l \leq r-1} \zeta^{-l} \zeta_1^l \zeta_n^{-l}.$$

In fact, this constitutes a presentation for H , as we will see in the remainder of this section. Constructing an H -module may therefore be done as follows: construct an H_{gr} -module together with operators Φ and Ψ satisfying the relations (3.1), (3.6), (3.7), and (3.8).

Theorem 3.1. *Let A be the algebra generated by H_{gr} together with elements Φ and Ψ satisfying (3.1), (3.6), (3.7), and (3.8). The natural map $A \rightarrow H$ is an isomorphism.*

Proof. Define elements $x_n, y_1 \in A$ by $x_n = \Phi s_1 \cdots s_{n-1}$ and $y_1 = \Psi s_{n-1} \cdots s_1$. Then put $x_i = w x_n w^{-1}$ and $y_i = v y_1 v^{-1}$ where $w, v \in S_n$ are chosen with $w(n) = i = v(1)$. The various x_i 's commute with one another by the discussion in 3.1, and by symmetry the y_i 's commute. Furthermore, the algebra A is generated by the group W together with x_1, \dots, x_n and y_1, \dots, y_n . We will show that it is spanned by the set of all words $x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} w$ with $a_i, b_i \in \mathbb{Z}_{\geq 0}$ and $w \in W$. From this together with the PBW theorem for H it will follow that the natural map from A to H is an isomorphism.

It suffices to show that the span of the set of words as above is closed under left multiplication by x_i 's, y_i 's, and w 's. This is clear for x_i 's and easy for w 's. We will show how to reorder a product $y_i x_j$. First observe that by the definitions of x_1, y_1 and the relation $\Psi\Phi = z_1$,

$$y_1 x_1 = \Psi\Phi = z_1.$$

By using the graded Hecke algebra relations between z_i and s_i it follows by induction on i that

$$z_i = y_i x_i + a_i \quad \text{for some } a_i \in \mathbb{C}W.$$

In particular

$$(3.9) \quad y_n x_n + a_n = z_n = \Phi\Psi + \kappa - \sum (d_j - d_{j-1})e_{nj} = x_n y_n + \kappa - \sum (d_j - d_{j-1})e_{nj}.$$

This proves that $y_n x_n = x_n y_n + b_n$ for some $b_n \in \mathbb{C}W$. Conjugating by some $w \in S_n$ with $w(n) = i$ gives $y_i x_i = x_i y_i + b_i$ for some $b_i \in \mathbb{C}W$. Using the last relation in (3.8), the relations (3.1) and (3.6) and the definitions of $y_1 = \Psi s_{n-1} \cdots s_1$ and $x_n = \Phi s_1 \cdots s_{n-1}$ allows one to rewrite $y_1 x_n = x_n y_1 + b_{1n}$ for some $b_{1n} \in \mathbb{C}W$, and conjugating by $w \in S_n$ with $w(1) = i$ and $w(n) = j$ gives $y_i x_j = x_j y_i + b_{ij}$ for some $b_{ij} \in \mathbb{C}W$, finishing the proof. \square

4. SPECHT-VALUED JACK POLYNOMIALS

For $\mu, \nu \in \mathbb{Z}_{\geq 0}^n$, write $\mu > \nu$ if either $\mu_+ >_d \nu_+$, where μ_+ and ν_+ are the partition rearrangements of μ and ν and $>_d$ denotes dominance order, or $\mu_+ = \nu_+$ and $w_\mu > w_\nu$ in Bruhat order. Extend this to a partial order on pairs (μ, T) by ignoring T : thus $(\mu, T) \geq (\nu, S)$ exactly if $\mu \geq \nu$. The following is Theorem 5.1 of [Gri2]; the polynomials it constructs are S^{λ^\bullet} -valued generalizations of non-symmetric Jack polynomials.

Theorem 4.1. *Let λ be an r -partition of n , $\mu \in \mathbb{Z}_{\geq 0}^n$, and let T be a standard tableau on λ . Put $v_T^\mu = w_\mu^{-1} \cdot v_T$ and recall the definitions of β and ct given in 2.10.*

(a) *The action of ζ_i and z_i on $\Delta(\lambda^\bullet)$ are given by*

$$\zeta_i \cdot x^\mu v_T^\mu = \zeta^{\beta(T^{-1}(w_\mu(i))) - \mu_i} x^\mu v_T^\mu$$

and

$$\begin{aligned} z_i \cdot x^\mu v_T^\mu &= (\mu_i + 1 - (d_{\beta(T^{-1}(w_\mu(i)))} - d_{\beta(T^{-1}(w_\mu(i))) - \mu_i - 1} - c_0 \text{rect}(T^{-1}(w_\mu(i)))) x^\mu v_T^\mu \\ &\quad + \sum_{(\nu, S) < (\mu, T)} c_{\nu, S} x^\nu v_S^\nu. \end{aligned}$$

(b) *Assuming that scalars are extended to $F = \mathbb{C}(c_0, d_1, d_2, \dots, d_{r-1})$, for each $\mu \in \mathbb{Z}_{\geq 0}^n$ and $T \in \text{SYT}(\lambda)$ there exists a unique \mathfrak{t} eigenvector $f_{\mu, T} \in \Delta(\lambda^\bullet)$ such that*

$$f_{\mu, T} = x^\mu v_T^\mu + \text{lower terms.}$$

The \mathfrak{t} -eigenvalue of $f_{\mu, T}$ is determined by the formulas in part (a).

We will also index these non-symmetric Jack polynomials $f_{P, Q}$, with $(P, Q) \in \Gamma$ as in 2.13.

4.1. Intertwiners. The intertwiners σ_i are defined, for $1 \leq i \leq n-1$, by

$$(4.1) \quad \sigma_i = s_i + \frac{c_0}{z_i - z_{i+1}} \pi_i.$$

Thus σ_i is well-defined on any \mathfrak{t} -weight space on which π_i acts by 0 or on which z_i and z_{i+1} have distinct eigenvalues.

For convenience, we reproduce here Lemma 5.3 of [Gri2], which describes how the intertwiners act on the basis $f_{\mu, T}$ of $\Delta(\lambda^\bullet)$. For $\mu \in \mathbb{Z}^n$ define

$$(4.2) \quad \phi(\mu_1, \dots, \mu_n) = (\mu_2, \mu_3, \dots, \mu_n, \mu_1 + 1) \quad \text{and} \quad \psi(\mu_1, \dots, \mu_n) = \phi^{-1}(\mu_1, \dots, \mu_n).$$

Lemma 4.2. *Let $\mu \in \mathbb{Z}_{\geq 0}^n$ and let T be a standard Young tableau on λ .*

(a) *Suppose $\mu_i \neq \mu_{i+1}$. If $\mu_i < \mu_{i+1}$ or $\mu_i - \mu_{i+1} \neq \beta(T^{-1}(w_\mu(i))) - \beta(T^{-1}(w_\mu(i+1))) \pmod r$ then*

$$\sigma_i \cdot f_{\mu, T} = f_{s_i \cdot \mu, T}.$$

(b) *If $\mu_i > \mu_{i+1}$ and $\mu_i - \mu_{i+1} = \beta(T^{-1}(w_\mu(i))) - \beta(T^{-1}(w_\mu(i+1))) \pmod r$ then*

$$\sigma_i \cdot f_{\mu, T} = \frac{(\delta - rc_0)(\delta + rc_0)}{\delta^2} f_{s_i \cdot \mu, T},$$

where

$$\delta = \kappa(\mu_i - \mu_{i+1}) - (d_{\beta(T^{-1}(w_\mu(i)))} - d_{\beta(T^{-1}(w_\mu(i+1)))}) - c_0 r (ct(T^{-1}(w_\mu(i))) - ct(T^{-1}(w_\mu(i+1)))).$$

(c) *Put $j = w_\mu(i)$. If $\mu_i = \mu_{i+1}$ then*

$$\sigma_i \cdot f_{\mu, T} = \begin{cases} 0 & \text{if } s_{j-1} \cdot T \text{ is not a standard tableau,} \\ f_{\mu, s_{j-1} \cdot T} & \text{if } \zeta^{\beta(T(j))} \neq \zeta^{\beta(T(j-1))}, \\ \left(1 - \left(\frac{1}{ct(T(j-1)) - ct(T(j))}\right)^2\right)^{1/2} f_{\mu, s_{j-1} \cdot T} & \text{else.} \end{cases}$$

(d) *For all $\mu \in \mathbb{Z}_{\geq 0}^n$,*

$$\Phi \cdot f_{\mu, T} = f_{\phi \cdot \mu, T}.$$

(e) *For all $\mu \in \mathbb{Z}_{\geq 0}^n$,*

$$\Psi \cdot f_{\mu, T} = \begin{cases} (\kappa \mu_n - (d_{\beta(T^{-1}(w_\mu(n)))} - d_{\beta(T^{-1}(w_\mu(n)) - \mu_n)} - r ct(T^{-1}(w_\mu(n))) c_0) f_{\psi \cdot \mu, T} & \text{if } \mu_n > 0, \\ 0 & \text{if } \mu_n = 0. \end{cases}$$

5. DIAGONALIZABILITY

5.1. Weight spaces. Fix an r -partition λ^\bullet and let $\Gamma = \mathbb{Z}_{\geq 0} \times \text{SYT}(\lambda^\bullet)$ (via the bijection of 2.13 this is the same Γ as defined there). Given $c = (c_0, d_0, \dots, d_{r-1}) \in \mathbb{C}^{r+1}$, $(\mu, T) \in \Gamma$ and $1 \leq i \leq n$, write $\text{wt}_c(\mu, T)_i$ for the pair

$$(5.1) \quad \text{wt}_c(\mu, T)_i = (\mu_i + 1 - (d_{\beta(T^{-1}w_\mu(i))} - d_{\beta(T^{-1}w_\mu(i)) - \mu_i - 1}) - r ct(T^{-1}w_\mu(i)) c_0, \zeta^{\beta(T^{-1}w_\mu(i)) - \mu_i})$$

Then define (μ, T) to be c -folded (or simply *folded* when c is fixed or clear from context) if $\text{wt}_c(\mu, T)_i = \text{wt}_c(\mu, T)_{i+1}$ for some $1 \leq i \leq n-1$. Foldings create non-trivial Jordan blocks:

Lemma 5.1. *Suppose that $\text{wt}_c(\mu, T)_i = \text{wt}_c(\mu, T)_{i+1}$ and write $\alpha = \text{wt}_c(\mu, T)_i$. Put $f_1 = f_{\mu, T}$ and $f_2 = s_i f_1$. If $z_i f_1 = \alpha f_1 = z_{i+1} f_1$, then $(z_i - \alpha) f_2 = -rc_0 f_1$ and $(z_{i+1} - \alpha) f_2 = rc_0 f_1$.*

Proof. Apply (3.5). □

As in [Gri2], for a box $b \in \lambda^\bullet$ and a positive integer k define a set

$$(5.2) \quad \Gamma_{b,k} = \{(\mu, T) \in \Gamma \mid \mu_{T(b)}^- \geq k\},$$

and for an ordered pair of distinct boxes $b_1, b_2 \in \Gamma$ and a positive integer $k \in \mathbb{Z}_{>0}$, define the subset $\Gamma_{b_1, b_2, k}$ of Γ by

$$(5.3) \quad \begin{aligned} (\mu, T) \in \Gamma_{b_1, b_2, k} & \iff \text{either } \mu_{T(b_1)}^- - \mu_{T(b_2)}^- > k \\ & \text{or } \mu_{T(b_1)}^- - \mu_{T(b_2)}^- = k \text{ and } w_\mu^{-1}(T(b_1)) < w_\mu^{-1}(T(b_2)). \end{aligned}$$

Via the bijection with pairs (P, Q) as above, these definitions become somewhat easier on the eyes:

$$(5.4) \quad \Gamma_{b,k} = \{(P, Q) \mid Q(b) \geq k\}$$

and

$$(5.5) \quad \Gamma_{b_1, b_2, k} = \{(P, Q) \mid Q(b_1) - Q(b_2) > k, \text{ or } Q(b_1) - Q(b_2) = k \text{ and } P(b_1) < P(b_2)\}$$

For a given parameter c , define the set $\Gamma_c \subseteq \Gamma$ by

$$(5.6) \quad \Gamma_c = \bigcap_{b,k} \Gamma_{b,k}^c \cap \bigcap_{b_1,b_2,k} \Gamma_{b_1,b_2,k}^c,$$

where for a subset $X \subseteq \Gamma$ we write X^c for its complement, the first intersection runs over pairs $b \in \lambda^\bullet$ and $k \in \mathbb{Z}_{>0}$ such that

$$k = d_{\beta(b)} - d_{\beta(b)-k} + r \text{ct}(b) c_0$$

and the second intersection runs over triples $b_1, b_2 \in \lambda^\bullet$, $k \in \mathbb{Z}_{>0}$ such that $k = \beta(b_1) - \beta(b_2) \bmod r$ and

$$k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\text{ct}(b_1) - \text{ct}(b_2) \pm 1) c_0.$$

The motivation for the definition is that the set Γ_c contains exactly those (μ, T) such that $f_{\mu,T}$ may be constructed from some $v_T \in S^{\lambda^\bullet}$ by applying a sequence of *invertible* intertwining operators; this is a consequence of Lemma 7.4 of [Gri2].

The definition of Γ_c may be rephrased in terms of pairs (P, Q) as follows: a pair (P, Q) is in Γ_c if and only if the following conditions hold:

- (a) whenever $b \in \lambda^\bullet$ and $k \in \mathbb{Z}_{>0}$ with $k = d_{\beta(b)} - d_{\beta(b)-k} + r \text{ct}(b) c_0$ we have $Q(b) < k$, and
- (b) whenever $b_1, b_2 \in \lambda^\bullet$ and $k \in \mathbb{Z}_{>0}$ with $\beta(b_1) - \beta(b_2) = k \bmod r$ and $k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\text{ct}(b_1) - \text{ct}(b_2) \pm 1) c_0$ we have $Q(b_1) \leq Q(b_2) + k$, with equality implying $P(b_1) > P(b_2)$.

The *boundary* of Γ_c is

$$(5.7) \quad \partial \Gamma_c = \{(\mu, T) \in \Gamma - \Gamma_c \mid (\psi \cdot \mu, T) \in \Gamma_c \text{ or } (s_i \cdot \mu, T) \in \Gamma_c \text{ for some } 1 \leq i \leq n\}.$$

Lemma 5.2. *Assume $c_0 \neq 0$.*

- (a) *Suppose $(\mu, T) \in \Gamma_c$ and $(\nu, S) \in \Gamma$ with $\text{wt}_c(\mu, T) = \text{wt}_c(\nu, S)$. Then $(\nu, S) = (\mu, T)$.*
- (b) *For all $(\mu, T) \in \Gamma_c$ and $1 \leq i \leq n-1$, we have $\text{wt}_c(\mu, T)_i \neq \text{wt}_c(\mu, T)_{i+1}$.*
- (c) *The non-symmetric generalized Jack polynomials $f_{\mu,T}$ for $(\mu, T) \in \Gamma_c \cup \partial \Gamma_c$ are all well-defined at c .*
- (d) *If $(\mu, T) \in \partial \Gamma_c$ is folded then $L_c(\lambda^\bullet)$ is not \mathfrak{t} -diagonalizable.*

Proof. The definition of Γ_c and Lemma 7.4 of [Gri2] together imply that the intertwiners connecting different \mathfrak{t} -weight spaces indexed by Γ_c are all invertible; it follows that every such weight space has the same dimension. Since the weight spaces in degree 0 (coming from S^{λ^\bullet}) are all one dimensional (here we use $c_0 \neq 0$), this proves (a). Part (b) follows from (a) together with Lemma 5.1. By part (b) the intertwining operators are well-defined on all weight spaces coming from Γ_c ; this allows one to recursively construct all Jack polynomials coming from $\Gamma_c \cup \partial \Gamma_c$ recursively, proving (c).

Now we prove (d). Suppose $(\mu, T) \in \partial \Gamma_c$ is folded and let $f_1 = f_{\mu,T}$. If $\text{wt}_c(\mu, T)_i = \text{wt}_c(\mu, T)_{i+1}$ then part (b) implies $z_i f = \alpha f = z_{i+1} f$ for some $\alpha \in \mathbb{C}$, and by Lemma 5.1 $f_2 = s_i f_1$ witnesses a non-trivial Jordan block for \mathfrak{t} : $(z_i - \alpha) f_2 = -r c_0 f_1 = -(z_{i+1} - \alpha) f_2$. We will show that the image of f_2 in $L_c(\lambda^\bullet)$ is non-zero. By (b) we must have either $(s_{i-1} \mu, T) \in \Gamma_c$ or $(s_{i+1} \mu, T) \in \Gamma_c$ or $(\psi \mu, T) \in \Gamma_c$.

Suppose that $(\psi \mu, T) \in \Gamma_c$. It follows that the map Ψ is not an injection on the weight space for (μ, T) , and hence by the second equation in (3.8) the z_n -eigenvalue on (μ, T) is given by $\alpha = 1 - (d_{\beta_j} - d_{\beta_{j-1}})$ where $\zeta_n f_{\mu,T} = \zeta^{\beta_j} f_{\mu,T}$. Compute using (3.8) and Lemma 5.1

$$\Phi \Psi f_2 = (z_n - 1 + \sum_{j=0}^{r-1} (d_j - d_{j-1}) e_{nj}) f_2 = (z_n - \alpha) f_2 = r c_0 f_1.$$

This equation implies that $\Psi f_2 = a f_{(\psi \mu, T)}$ for some $a \in \mathbb{C}^\times$ and since the image of $f_{(\psi \mu, T)}$ in $L_c(\lambda^\bullet)$ is non-zero, so is the image of f_2 . \square

Now we can give our first (not completely explicit) description of the diagonalizable $L_c(\lambda^\bullet)$'s. When $c_0 = 0$ the modules $\Delta_c(\lambda^\bullet)$ are all diagonalizable (but with weight spaces of dimension greater than 1), so the following theorem finishes the classification.

Theorem 5.3. *Suppose $c_0 \neq 0$. The module $L_c(\lambda^\bullet)$ is diagonalizable exactly if no element of $\partial\Gamma_c$ is folded; in this case a basis is given by $\{f_{\mu,T} \mid (\mu,T) \in \Gamma_c\}$.*

Proof. Given Lemma 5.2, it remains to show that if no element of $\partial\Gamma_c$ is folded then $L_c(\lambda^\bullet)$ is diagonalizable with the given basis. Let V be the abstract \mathbb{C} -vector space with basis given by $\{f_{\mu,T} \mid (\mu,T) \in \Gamma_c\}$, and define actions of \mathfrak{t} , $\sigma_1, \dots, \sigma_{n-1}$, Φ , and Ψ on V as follows: the \mathfrak{t} -action has $f_{\mu,T}$ as eigenfunctions with eigenvalues given by Theorem 4.1, and the action of σ_i , Φ , and Ψ is given by the formulas in Lemma 4.2 with the following exceptions: if $(s_i\mu, T) \notin \Gamma_c$ then we put $\sigma_i f_{\mu,T} = 0$ and if $(\phi\mu, T) \notin \Gamma_c$ then we put $\Phi f_{\mu,T} = 0$. Without using the hypothesis that no element of $\partial\Gamma_c$ is folded, it follows from these definitions that the σ_i 's satisfy the braid relations, that

$$(5.8) \quad \sigma_i^2 = \frac{(z_i - z_{i+1} - c_0\pi_i)(z_i - z_{i+1} + c_0\pi_i)}{(z_i - z_{i+1})^2},$$

that

$$(5.9) \quad f\sigma_i = \sigma_i s_i(f) \quad \text{for } 1 \leq i \leq n-1 \text{ and } f \in \mathfrak{t},$$

that

$$(5.10) \quad \Phi\sigma_i = \sigma_{i-1}\Phi \quad \text{for } 2 \leq i \leq n-1 \text{ and } \Phi^2\sigma_1 = \sigma_{n-1}\Phi^2,$$

that

$$(5.11) \quad \Psi\sigma_i = \sigma_{i+1}\Psi \quad \text{for } 1 \leq i \leq n-2 \text{ and } \Psi^2 s_{n-1} = s_1\Psi^2,$$

that

$$(5.12) \quad f\Phi = \Phi\phi(f) \quad \text{and} \quad f\Psi = \Psi\phi^{-1}(f) \quad \text{for } f \in \mathfrak{t}$$

and that

$$(5.13) \quad \Psi\Phi = z_1, \quad \Phi\Psi = z_n - \kappa + \sum_{j=0}^{r-1} (d_j - d_{j-1})e_{nj}, \quad \text{and} \quad \Psi\sigma_{n-1}\Phi = \Phi\sigma_1\Psi.$$

Indeed, these formulas hold by Lemma 4.2 when applied to those $f_{\mu,T}$ for which the result stays in Γ_c at each stage; some care must be taken near the boundary, as we indicate next.

We will sketch a check of the relation $\Psi\sigma_{n-1}\Phi = \Phi\sigma_1\Psi$ here; the others involve similar reasoning. Suppose first that $\Phi f_{P,Q} = 0$, that is, $\phi(P,Q) \notin \Gamma_c$. By definition of Γ_c and ϕ , there is a box b with $Q(b) = k-1$, $P(b) = 1$ and such that if $Q'(b) \geq k$ for some $(P', Q') \in \Gamma$ then $(P', Q') \notin \Gamma_c$. Now observe that $\phi s_1\psi(P, Q) = (P', Q')$ with $Q'(b) = k$ and hence $(P', Q') \notin \Gamma_c$, so both operators act by zero on $f_{P,Q}$.

If $\phi(P, Q) \in \Gamma_c$ but $s_{n-1}\phi(P, Q) \notin \Gamma_c$, then writing $(P', Q') = \phi(P, Q)$, setting $b_1 = P'^{-1}(n)$ and $b_2 = P'^{-1}(n-1)$ we have $Q'(b_1) = Q'(b_2) + k$ for some positive integer k , $k = \beta(b_1) - \beta(b_2) \bmod r$, such that if $(P_0, Q_0) \in \Gamma$ with $Q_0(b_1) > Q_0(b_2) + k$, or $Q_0(b_1) = Q_0(b_2) + k$ and $P_0(b_1) < P_0(b_2)$, then $(P_0, Q_0) \notin \Gamma_c$. It follows from this that $s_1\psi(P, Q) \notin \Gamma_c$, so again both operators act by zero on $f_{P,Q}$. The other cases are handled in a similar fashion.

Now define the action of s_1, \dots, s_{n-1} on V by the formula

$$(5.14) \quad s_i = \sigma_i - \frac{c_0}{z_i - z_{i+1}} \pi_i.$$

This makes sense by part (b) of Lemma 5.2. Using Theorem 3.1 we must check that the s_i 's and \mathfrak{t} satisfy the graded Hecke relations. The relations (3.5) follow from the definition (5.14) and the

relations (5.9). The fact that $s_i^2 = 1$ follows from (5.14) and (5.8). The fact that the braid relations are satisfied will be the first place the hypothesis that $\partial\Gamma_c$ contains no folds is used. Compute:

$$\begin{aligned}
s_i s_{i+1} s_i &= \left(\sigma_i - \frac{c_0}{z_i - z_{i+1}} \pi_i \right) \left(\sigma_{i+1} - \frac{c_0}{z_{i+1} - z_{i+2}} \pi_{i+1} \right) \left(\sigma_i - \frac{c_0}{z_i - z_{i+1}} \pi_i \right) \\
&= \sigma_i \sigma_{i+1} \sigma_i - c_0 \sigma_i \sigma_{i+1} \frac{1}{z_i - z_{i+1}} \pi_i - c_0 \sigma_i^2 \frac{1}{z_i - z_{i+2}} \pi_{i,i+2} - c_0 \sigma_{i+1} \sigma_i \frac{1}{z_{i+1} - z_{i+2}} \pi_{i+1} \\
&\quad + c_0^2 \sigma_i \frac{1}{(z_{i+1} - z_{i+2})(z_i - z_{i+1})} \pi_i \pi_{i+1} + c_0^2 \sigma_{i+1} \frac{1}{(z_i - z_{i+2})(z_i - z_{i+1})} \pi_{i,i+2} \pi_{i+1} \\
&\quad + c_0^2 \sigma_i \frac{1}{(z_{i+1} - z_i)(z_i - z_{i+2})} \pi_i \pi_{i,i+2} - c_0^3 \frac{1}{(z_i - z_{i+1})^2 (z_{i+1} - z_{i+2})} \pi_i \pi_{i+1} \pi_i.
\end{aligned}$$

This preceding calculation was formal, but the hypothesis that no element of $\partial\Gamma_c$ is folded is exactly what is needed to ensure that the right-hand side of the above equation is well-defined when applied to $f_{\mu,T}$ for all $(\mu, T) \in \Gamma_c$. Routine arithmetic verifies that it is the same as the corresponding expression for $s_{i+1} s_i s_{i+1}$. This verifies that we have the structure of an \mathbb{H}_{gr} -module on V .

Verification of the relations (3.1), (3.6), and the last equation in (3.8) is exactly analogous, again using the hypothesis that no element of $\partial\Gamma_c$ is folded: for instance, one computes

$$\begin{aligned}
(5.15) \quad \Psi s_{n-1} \Phi &= \Psi \left(\sigma_{n-1} - \frac{c_0}{z_{n-1} - z_n} \pi_{n-1,n} \right) \Phi = \Phi \sigma_1 \Psi - \Psi \Phi \phi \left(\frac{c_0}{z_{n-1} - z_n} \pi_{n-1,n} \right) \\
&= \Phi s_1 \Psi - \Phi \Psi \phi^{-1} \left(\frac{c_0}{z_1 - z_2} \pi_{1,2} \right) - \Psi \Phi \phi \left(\frac{c_0}{z_{n-1} - z_n} \pi_{n-1,n} \right).
\end{aligned}$$

The hypothesis that there are no folded elements of $\partial\Gamma_c$ implies that this last expression makes sense when applied to any $f_{P,Q}$ for $(P, Q) \in \Gamma_c$, and a straightforward calculation shows that it is equivalent to the last relation in (3.8). \square

6. COMBINATORICS OF FOLDS

6.1. Near folds. We first obtain some limitations on the types of folds that can occur in $\partial\Gamma_c$. First, we switch from now on to the (P, Q) notation for elements of Γ , and we define a *near fold* to be an element $(P, Q) \in \Gamma_c$ such that $\phi(P, Q)$ or $s_i(P, Q)$ is folded for some $1 \leq i \leq n-1$ (by Lemma 5.2 this fold is then in the boundary $\partial\Gamma_c$). Here we define $s_i(P, Q) = (s_i P, Q)$, and $\phi(P, Q) = (P', Q')$ with

$$(6.1) \quad P'(b) = \begin{cases} P(b) - 1 & \text{if } P(b) > 1, \text{ and} \\ n & \text{if } P(b) = 1, \end{cases}$$

and

$$(6.2) \quad Q'(b) = \begin{cases} Q(b) & \text{if } P(b) > 1, \text{ and} \\ Q(b) + 1 & \text{if } P(b) = 1. \end{cases}$$

These definitions are compatible with the corresponding ones for the (μ, T) notation via the bijection of 2.13.

The *upper rim* of a partition λ is the set of boxes $b \in \lambda$ such that there is no box immediately above b . The *upper rim* of an r -partition λ^\bullet is the union of the upper rims of its components λ^l . The *left rim* of a partition (resp. multipartition) is defined analogously as the set of boxes with no box immediately to the left.

Lemma 6.1. $(P, Q) \in \Gamma_c$ is a near fold if and only if there is a positive integer k and boxes $b_1, b_2 \in \lambda^\bullet$ such that b_1 is a removable box, b_2 is on the upper rim or left rim of λ^\bullet , $k = \beta(b_1) - \beta(b_2) \bmod r$,

$$k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\text{ct}(b_1) - \text{ct}(b_2))c_0,$$

and either

- (a) $\text{ct}(b_2) = 0$ (so b_2 is the upper left hand corner of $\lambda^{\beta(b_2)}$), $Q(b_1) = k - 1$, $Q(b_2) = 0$, $P(b_1) = 1$, and $P(b_2) = n$, or
- (b) $\text{ct}(b_2) \neq 0$ and there are $a \in \mathbb{Z}_{\geq 0}$ and a box $b_3 < b_2$ adjacent to b_2 with $\text{ct}(b_3) = \text{ct}(b_2) \pm 1$ with $Q(b_1) = a + k$, $Q(b_2) = a = Q(b_3)$, $P(b_1) = i + 1$, $P(b_2) = i - 1$, and $P(b_3) = i$ for some $2 \leq i \leq n - 1$.

Proof. First, if (a) holds then $\phi(P, Q)$ is folded, and if (b) holds then $s_i(P, Q)$ is folded. This follows from the formula given in 2.13 for the t -eigenvalue of $f_{P, Q}$, together with the formulas for $\phi(P, Q)$ and $s_i(P, Q)$ given above.

For the converse, assume first that $s_i(P, Q) = (s_i P, Q)$ is folded for some $(P, Q) \in \Gamma_c$ and $1 \leq i \leq n - 1$. We will show that in this case we are in situation (b) of the lemma. It follows from our assumption that either

$$Q(P^{-1}(i)) - Q(P^{-1}(i + 2)) = d_{\beta(P^{-1}(i))} - d_{\beta(P^{-1}(i+2))} + r(\text{ct}(P^{-1}(i)) - \text{ct}(P^{-1}(i + 2)))c_0$$

with $Q(P^{-1}(i)) - Q(P^{-1}(i + 2)) = \beta(P^{-1}(i)) - \beta(P^{-1}(i + 2)) \bmod r$, or that the analogous equations, replacing i and $i + 2$ by $i - 1$ and $i + 1$, hold. In any case there are boxes $b_1, b_2 \in \lambda^\bullet$ and a non-negative integer k with

$$(6.3) \quad k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\text{ct}(b_1) - \text{ct}(b_2))c_0,$$

$$k = \beta(b_1) - \beta(b_2) \bmod r, \quad Q(b_1) - Q(b_2) = k, \quad \text{and} \quad |P(b_1) - P(b_2)| = 2.$$

First observe that $k > 0$ since otherwise $Q(b_1) = Q(b_2)$, $\beta(b_1) = \beta(b_2)$, and $\text{ct}(b_1) = \text{ct}(b_2)$ contradicts $|P(b_1) - P(b_2)| = 2$. Since $(P, Q) \in \Gamma_c$, (6.3) implies $\text{ct}(b_2) \neq 0$. If b_2 is not on the upper or left rim of λ^\bullet , there are two boxes $b_3, b_4 < b_2$ with $\text{ct}(b_3) = \text{ct}(b_2) \pm 1$ and $\text{ct}(b_4) = \text{ct}(b_2) \pm 1$. Now (6.3) together with $(P, Q) \in \Gamma_c$ implies that $Q(b_3) = Q(b_2) = Q(b_4)$ and that $P(b_1) > P(b_3), P(b_4) > P(b_2)$, contradicting $|P(b_1) - P(b_2)| = 2$. On the other hand, since $\text{ct}(b_2) \neq 0$ there is always at least one box b_3 as above, so we have $P(b_1) = P(b_3) + 1 = P(b_2) + 2$.

If b_1 is not a removable box, then there is a box $b > b_1$ such that $\text{ct}(b) = \text{ct}(b_1) \pm 1$, and (6.3) once more implies $Q(b) = Q(b_1)$ and hence $P(b_1) > P(b) > P(b_2)$, which contradicts $P(b_1) = P(b_3) + 1 = P(b_2) + 2$.

Now assume $\phi(P, Q)$ is folded for some $(P, Q) \in \Gamma_c$. Then

$$Q(P^{-1}(1)) + 1 - Q(P^{-1}(n)) = d_{\beta(P^{-1}(1))} - d_{\beta(P^{-1}(n))} + r(\text{ct}(P^{-1}(1)) - \text{ct}(P^{-1}(n)))c_0$$

and

$$Q(P^{-1}(1)) + 1 - Q(P^{-1}(n)) = \beta(P^{-1}(1)) - \beta(P^{-1}(n)) \bmod r.$$

Assume first that $Q(P^{-1}(1)) + 1 - Q(P^{-1}(n)) < 0$ and let $k = Q(P^{-1}(n)) - Q(P^{-1}(1)) - 1$, $b_1 = P^{-1}(n)$ and $b_2 = P^{-1}(1)$. If b_2 is the upper left hand corner of $\lambda^{\beta(b_2)}$ then $\text{ct}(b_2) = 0$ and the equations

$$k = d_{\beta(b_1)} - d_{\beta(b_2)} + r\text{ct}(b_1)c_0 \quad \text{and} \quad k = \beta(b_1) - \beta(b_2) \bmod r$$

together with $(P, Q) \in \Gamma_c$ force $Q(b_1) < k = Q(b_1) - Q(b_2) - 1$, contradiction. Thus b_2 is not the upper left-hand corner of $\lambda^{\beta(b_2)}$ and hence there is a box $b_3 < b_2$ with $\text{ct}(b_3) = \text{ct}(b_2) \pm 1$. Now

$$k = d_{\beta(b_1)} - d_{\beta(b_3)} + r(\text{ct}(b_1) - \text{ct}(b_3) \pm 1)c_0 \quad \text{and} \quad k = \beta(b_1) - \beta(b_3) \bmod r$$

together with $(P, Q) \in \Gamma_c$ imply that $Q(b_1) - Q(b_3) \leq k = Q(b_1) - Q(b_2) - 1 \leq Q(b_1) - Q(b_3) - 1$, contradiction. Therefore $Q(P^{-1}(n)) - Q(P^{-1}(1)) - 1 \geq 0$.

Let $k = Q(P^{-1}(1)) + 1 - Q(P^{-1}(n))$, $b_1 = P^{-1}(n)$ and $b_2 = P^{-1}(1)$. Then $k \geq 0$.

If $k = 0$ then $\beta(b_1) = \beta(b_2)$, $Q(b_1) = Q(b_2) + 1$, $\text{ct}(b_1) = \text{ct}(b_2)$ and $Q(b_1) = Q(b_2) - 1$ implying $b_1 < b_2$. But since $P(b_1) = 1$ there can be no box $b > b_1$ with $Q(b) = Q(b_1)$ and likewise since $P(b_2) = n$ there can be no box $b < b_2$ with $Q(b) = Q(b_2)$, contradiction.

Therefore $k > 0$, and the equations

$$k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\text{ct}(b_1) - \text{ct}(b_2))c_0 \quad \text{and} \quad k = \beta(b_1) - \beta(b_2) \bmod r$$

together with $(P, Q) \in \Gamma_c$, $P(b_1) = 1$ and $P(b_2) = n$ imply that there is no box b with $b > b_1$ or $b < b_2$, and hence also that $Q(b_1) = k - 1$. This proves that we are in case (a) of the lemma. \square

6.2. Proof of Theorem 2.1. Thanks to Theorem 5.3, the Theorem 2.1 is a consequence of the following:

Theorem 6.2. *Assume $c_0 \neq 0$. The module $L_c(\lambda^\bullet)$ is diagonalizable if and only if for every removable box $b \in \lambda^\bullet$, either $k_c(b) = \infty$ or $l_c(b) < k_c(b)$.*

Proof. For convenience, we define a statistic $l'_c(b)$ similar to $l_c(b)$ (as in 2.14), but without restricting to outside addable boxes: more precisely, $l'_c(b)$ is the smallest integer l such that either there exists a box $b' \in \lambda^{\beta(b)-l}$ with

$$l = d_{\beta(b)} - d_{\beta(b)-l} + r(\text{ct}(b) - \text{ct}(b') \pm 1)c_0$$

or

$$l = d_{\beta(b)} - d_{\beta(b)-l} + r\text{ct}(b)c_0.$$

We will prove the theorem first replacing $l_c(b)$ with $l'_c(b)$; it is easy to see that the theorem follows from this.

Suppose first that there is a removable box $b_1 \in \lambda^\bullet$ with $k_c(b_1) < \infty$ and $k_c(b_1) \leq l'_c(b_1)$ and write $k = k_c(b_1)$. Then there is b_2 in the upper or left rim of $\lambda^{\beta(b_1)-k}$ with

$$k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\text{ct}(b_1) - \text{ct}(b_2))c_0.$$

If b_2 is the upper left-hand corner of $\lambda^{\beta(b_1)-k}$ then set $Q(b_1) = k - 1$ and $Q(b) = 0$ for all other $b \in \lambda^\bullet$, and put $P(b_1) = 1$, $P(b_2) = n$ and then complete P to a reverse standard Young tableau on λ^\bullet . It follows from the assumption $k_c(b_1) \leq l'_c(b_1)$ that all the inequalities necessary for $(P, Q) \in \Gamma_c$ are satisfied, and therefore (P, Q) is a near fold in Γ_c so $L_c(\lambda^\bullet)$ is not diagonalizable.

If $\text{ct}(b_2) \neq 0$, then we define $Q(b_1) = k$ and $Q(b) = 0$ for all other $b \in \lambda^\bullet$. We now define P for which $(P, Q) \in \Gamma_c$: let b_3 be the box with $b_3 < b_2$ and $\text{ct}(b_3) = \text{ct}(b_2) \pm 1$, let i be maximal so that there exists a reverse standard Young tableau T on λ^\bullet with $T(b_2) = i$, and define $P(b_1) = i + 1$, $P(b_2) = i - 1$, $P(b_3) = i$ and then complete P to a reverse standard Young tableau on $\lambda^\bullet - \{b\}$. It is then straightforward to check that $(P, Q) \in \Gamma_c$ is a near fold so that $L_c(\lambda^\bullet)$ is not diagonalizable.

Conversely, suppose that $l'_c(b) < k_c(b)$ for all corner boxes b such that $k(b) < \infty$. By Theorem 5.3 it suffices to show that there are no near folds in Γ_c . Suppose towards a contradiction that $(P, Q) \in \Gamma_c$ is a near fold. By Lemma 6.1 there is a corner box $b_1 \in \lambda^\bullet$, an integer $k \in \mathbb{Z}_{>0}$, and a box b_2 in the upper or left rim of $\lambda^{\beta(b_1)-k}$ with

$$k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\text{ct}(b_1) - \text{ct}(b_2))c_0,$$

and one of the following holds:

- (a) The box b_2 is the upper left-hand corner of $\lambda^{\beta(b_1)-k}$, and $Q(b_1) = k - 1$, $Q(b_2) = 0$, $P(b_1) = 1$, and $P(b_2) = n$, or
- (b) there is a box $b_3 < b_2$ with $\text{ct}(b_3) = \text{ct}(b_2) \pm 1$ and so that $Q(b_1) = Q(b_2) + k = Q(b_3) + k$ and $P(b_1) = P(b_3) + 1 = P(b_2) + 2$.

Assume that case (b) holds; (a) is similar.

By hypothesis there is some integer $0 < l < k$ such that either $l = d_{\beta(b_1)} - d_{\beta(b_1)-l} + r\text{ct}(b_1)c_0$ (in which case $(P, Q) \notin \Gamma_c$) or so that there is a box $b_4 \in \lambda^{\beta(b_1)-l}$ with

$$l = d_{\beta(b_1)} - d_{\beta(b_4)} + r(\text{ct}(b_1) - \text{ct}(b_4) \pm 1)c_0$$

in which case also

$$0 < k - l = d_{\beta(b_4)} - d_{\beta(b_2)} + r(\text{ct}(b_4) - \text{ct}(b_2) \pm 1)c_0$$

and $k - l = \beta(b_4) - \beta(b_2) \bmod r$, implying that

- (1) $Q(b_1) \leq Q(b_4) + l$ with equality implying $P(b_1) > P(b_4)$ and
- (2) $Q(b_4) \leq Q(b_2) + k - l$ with equality implying $P(b_4) > P(b_2)$.

Observe first that if $b_4 = b_3$ then the requirement $Q(b_1) \leq Q(b_4) + l < Q(b_4) + k$ precludes $(P, Q) \in \Gamma_c$. Thus $b_4 \neq b_3$. On the other hand, combining (1) and (2) above shows that $Q(b_1) \leq Q(b_2) + k$ with equality implying $P(b_1) > P(b_4) > P(b_2)$, and this contradicts $P(b_1) = P(b_3) + 1 = P(b_2) + 2$. \square

7. PROOF OF THEOREM 2.4

7.1. The same argument as in the appendix of [EtSt] shows that $L_c(\lambda^\bullet)$ is unitary exactly if it is diagonalizable and for all $(P, Q) \in \Gamma_c$ we have

$$(7.1) \quad Q(P^{-1}(1)) + 1 \geq d_{\beta(P^{-1}(1))} - d_{\beta(P^{-1}(1)) - Q(P^{-1}(1)) - 1} + r\text{ct}(P^{-1}(1))c_0$$

and if $1 \leq i \leq n - 1$ with $Q(P^{-1}(i)) - Q(P^{-1}(i + 1)) = \beta(P^{-1}(i)) - \beta(P^{-1}(i + 1)) \bmod r$ then setting $b_1 = P^{-1}(i)$ and $b_2 = P^{-1}(i + 1)$

$$(7.2) \quad (Q(b_1) - Q(b_2) - (d_{\beta(b_1)} - d_{\beta(b_2)}) - r(\text{ct}(b_1) - \text{ct}(b_2))c_0)^2 \geq (rc_0)^2.$$

Assuming $c_0 > 0$ the last condition may be rephrased: $L_c(\lambda^\bullet)$ is not unitary if

$$(7.3) \quad d_{\beta(b_1)} - d_{\beta(b_2)} + r(\text{ct}(b_1) - \text{ct}(b_2) - 1)c_0 < Q(b_1) - Q(b_2) \\ < (d_{\beta(b_1)} - d_{\beta(b_2)}) + r(\text{ct}(b_1) - \text{ct}(b_2) + 1)c_0$$

for some $(P, Q) \in \Gamma_c$ with $P(b_1) = i$, $P(b_2) = i + 1$, and $Q(b_1) - Q(b_2) = \beta(b_1) - \beta(b_2) \bmod r$.

7.2. We suppose first that $c_0 > 0$ and that $L_c(\lambda^\bullet)$ is diagonalizable but that either

- (a) there exists a pair (b_1, b_2) of boxes of λ^\bullet such that, writing $i = \beta(b_1)$ and $j = \beta(b_2)$, we have

$$d_i - d_j + r(\text{ct}(b_1) - \text{ct}(b_2) + 1)c_0 > m_{ij} > d_i - d_j + r(\text{ct}(b_1) - \text{ct}(b_2) - 1)c_0$$

and there does not exist a blocking sequence for (b_1, b_2) , or

- (b) there exists a pair (b, j) consisting of a box $b \in \lambda^i$ and $0 \leq j \leq r - 1$ such that

$$d_i - d_j + r\text{ct}(b)c_0 > m_{ij}$$

and there does not exist a blocking sequence for (b, j) .

In the next subsection we will construct certain pairs (P, Q) that will violate unitarity.

7.3. **Construction of unitarity-preventing (P, Q) 's.** Suppose that (b_1, b_2) is a pair of boxes for which there is no blocking sequence and put $i = \beta(b_1)$ and $j = \beta(b_2)$. We will construct $(P, Q) \in \Gamma_c$ with $Q(b_1) = m_{ij}$, $Q(b_2) = 0$, $P(b_1) = a + 1$, and $P(b_2) = a$ for some $1 \leq a \leq n - 1$:

For each $b \geq b_1$ set $Q(b) = m_{ij}$, and for each $b \leq b_2$ set $Q(b) = 0$ (note that b_1, b_2 would be a blocking sequence if $b_1 \leq b_2$, so this first step is compatible with our aim). Define $P(b)$ for all $b > b_1$ and all $b < b_2$ in such a way that P is decreasing on these posets, and furthermore so that the set of numbers $P(b)$ thus defined is equal to the set $\{d, d + 1, \dots, n\}$, where $n - d + 1$ is the number of boxes b with $b \geq b_1$ or $b \leq b_2$ (this last condition will force $P(b) < d$ when we define $P(b)$ for the remaining boxes b of λ^\bullet).

Assuming we have defined Q and P on all boxes in $b \in \lambda^{i-1}, \lambda^{i-2}, \dots, \lambda^{i-k+1}$, we define them on λ^{i-k} by induction, choosing the minimal b for which Q and P are not already defined. Choose $P(b)$ maximal from among the unused numbers in $\{1, 2, \dots, n\}$. We choose $Q(b)$ minimal subject to the conditions:

- (a) $Q(b) \geq 0$,

- (b) $Q(b) \geq Q(b')$ for all $b' \leq b$, and
- (c) for each box b' such that $Q(b')$ and $P(b')$ have already been defined, and for any positive integer l with $l = \beta(b') - \beta(b) \bmod r$ and

$$l = d_{\beta(b')} - d_{\beta(b)} + r(\text{ct}(b') - \text{ct}(b) \pm 1)c_0,$$

we enforce $Q(b') - Q(b) \leq l$.

The procedure evidently has the property if P and Q are defined on b after they are defined on b' then $P(b) < P(b')$ (we regard the initial definitions for $b \geq b_1$ and $b' \leq b_2$ as happening simultaneously). Furthermore, one proves by induction on k that for $1 \leq k \leq m_{ij} - 1$ we have $Q(b) \leq m_{ij} - k$ for all $b \in \lambda^{i-k}$, and that $Q(b) = 0$ for all other b provided that $b \not\geq b_1$.

We check that $(P, Q) \in \Gamma$ first. Write X for the set of boxes of λ which are $\geq b_1$ or $\leq b_2$. If $b' < b$ are boxes of λ^\bullet not in X , then we defined Q and P on b' before b , so that $P(b') > P(b)$, and $Q(b) \leq Q(b')$ by (a) above. If $b \in X$ then either $b' < b \leq b_2$, in which case $Q(b) = Q(b') = 0$ and $P(b') > P(b)$ by construction, or $b \geq b_1$. If also $b' \geq b_1$ then by the first step in our construction we are done. Otherwise $Q(b') < m_{ij} = Q(b)$, which implies that $(P, Q) \in \Gamma$. The reasoning in case $b' \in X$ is similar.

Now we check that $(P, Q) \in \Gamma_c$. Let $b \in \lambda^\bullet$ and $l \in \mathbb{Z}_{>0}$ such that

$$d_{\beta(b)} - d_{\beta(b)-l} + r\text{ct}(b)c_0 = l.$$

We must check $Q(b) < l$. If $b \geq b_1$ then b_1, b is a blocking sequence unless $l > m_{ij} = Q(b)$. If $Q(b) = 0$ then obviously $Q(b) = 0 < l$, so now we may assume that $Q(b) > 0$. We choose $b' \leq b$ minimal such that $Q(b') = Q(b)$. By the procedure defining (P, Q) , there is an equation of the form

$$d_{\beta(b'')} - d_{\beta(b')} + r(\text{ct}(b'') - \text{ct}(b') \pm 1)c_0 = k$$

for some positive integer k and some box $b'' \in \lambda^{\beta(b')+k}$ such that $Q(b'') = Q(b') + k$. Repeating this reasoning with b'' in place of b gives a blocking sequence if $Q(b) \geq l$.

Likewise, if $b, b' \in \lambda^\bullet$ with

$$d_{\beta(b)} - d_{\beta(b')} + r(\text{ct}(b) - \text{ct}(b') \pm 1)c_0 = l$$

for a positive integer l with $l = \beta(b) - \beta(b') \bmod r$, then the construction of (P, Q) implies that $Q(b) - Q(b') \leq l$, and that equality can only occur if $l = m_{\beta(b), \beta(b')}$, $b \in \lambda^{i-k}$ and $b' \in \lambda^{i-k'}$ with $0 \leq k < k' \leq m_{ij}$. Furthermore, $P(b) > P(b')$ unless $b' \leq b_2$. So we may assume that $Q(b) = Q(b') + m_{\beta(b), \beta(b')}$ and that $b' \leq b_2$ (hence $Q(b) = m_{\beta(b), \beta(b')}$). As before, we can now construct a blocking sequence that ends with b, b', b_2 .

An exactly analogous construction shows that if $b \in \lambda^i$ and $0 \leq j \leq r - 1$ are such that there is no blocking sequence, then there is $(P, Q) \in \Gamma_c$ such that $Q(b) = m_{ij} - 1$ and $P(b) = 1$.

7.4. For the converse, if (7.3) holds then we certainly have

$$m_{ij} < d_{\beta(b_1)} - d_{\beta(b_2)} + r(\text{ct}(b_1) - \text{ct}(b_2) + 1)c_0,$$

and if $c_0 > 0$ then either we can find $b_3 \leq b_2$ with

$$d_{\beta(b_1)} - d_{\beta(b_3)} + r(\text{ct}(b_1) - \text{ct}(b_3) - 1)c_0 < m_{ij} < d_{\beta(b_1)} - d_{\beta(b_3)} + r(\text{ct}(b_1) - \text{ct}(b_3) + 1)c_0,$$

or we have

$$d_{\beta(b_1)} - d_{\beta(b_2)} + r\text{ct}(b_1)c_0 > m_{ij}.$$

In the first case a blocking sequence for (b_1, b_3) may be modified to give one for (b_1, b_2) , and in the second case a blocking sequence for $(b_1, \beta(b_2))$ is one for (b_1, b_2) . In either case the existence of the blocking sequence combined with Lemma 7.1 below implies that (P, Q) satisfying (7.3) cannot be in Γ_c . Likewise if (7.1) fails for $(P, Q) \in \Gamma_c$ then with $b = P^{-1}(1)$, $i = \beta(b)$, and $j = \beta(b) - Q(b) - 1$ we have

$$m_{ij} < d_i - d_j + r\text{ct}(b)c_0,$$

and hence there exists a blocking sequence for (b, j) , and Lemma 7.2 below implies that $(P, Q) \notin \Gamma_c$.

Lemma 7.1. *If $b \in \lambda^i$, $b' \in \lambda^j$, a blocking sequence for (b, b') exists, and $(P, Q) \in \Gamma_c$ with $Q(b) - Q(b') \geq m_{ij}$ and $P(b) = P(b') + 1$, then $Q(b) = Q(b') + m_{ij}$ and*

$$m_{ij} = d_i - d_j + r(\text{ct}(b) - \text{ct}(b') \pm 1)c_0.$$

Proof. Let $(b_0, b_1, \dots, b_{2q+1})$ be a blocking sequence. Since $(P, Q) \in \Gamma$, for $0 \leq k \leq q$ we have $Q(b_{2k}) \leq Q(b_{2k+1})$ with equality implying that $P(b_{2k}) > P(b_{2k+1})$ unless $b_{2k} = b_{2k+1}$. Since $(P, Q) \in \Gamma_c$, for $1 \leq k \leq q$ we have $Q(b_{2k-1}) \leq Q(b_{2k}) + m_{\beta(b_{2k-1}), \beta(b_{2k})}$ with equality implying $P(b_{2k-1}) > P(b_{2k})$. Combining these inequalities, we have

$$Q(b_0) \leq Q(b_{2q+1}) + \sum_{k=1}^q m_{\beta(b_{2k-1}), \beta(b_{2k})} \leq Q(b_{2q+1}) + m_{ij},$$

with equality implying $P(b_0) \geq P(b_{2q+1}) + d - 1$ where d is the number of distinct boxes that appear in the sequence $b_0, b_1, \dots, b_{2q+1}$. Now according to the definition of blocking sequence, there are two cases: first, if $b_{2q+1} = b'$, then since $P(b) = P(b') + 1$ we must have $d = 2$, whence $b_1 = b_0 = b$, $b_2 = b_3 = b'$, and

$$d_{\beta(b_1)} - d_{\beta(b_2)} + r(\text{ct}(b_1) - \text{ct}(b_2) \pm 1)c_0 = m_{ij}$$

so we are done. Second, if there is a positive integer l with

$$m_{\beta(b_{2q+1}), l} + \sum_{k=1}^q m_{\beta(b_{2k-1}), \beta(b_{2k})} \leq m_{ij} \quad \text{and} \quad d_{\beta(b_{2q+1})} - d_{\beta(b_{2q+1})-l} + r\text{ct}(b_{2q+1})c_0 = l,$$

then the preceding inequality combined with $Q(b_{2q+1}) \leq l - 1$ implies $Q(b) < m_{ij}$, contradicting $Q(b) \geq Q(b') + m_{ij}$. \square

Lemma 7.2. *If $b \in \lambda^i$ and $0 \leq j \leq r - 1$ is a pair for which a blocking sequence exists, and there is $(P, Q) \in \Gamma_c$ with $Q(b) \geq m_{ij} - 1$ and $P(b) = 1$ then we must have*

$$d_{\beta(b)} - d_j + r\text{ct}(b)c_0 = m_{ij}.$$

Proof. If $b = b_0, b_1, \dots, b_{2q+1}$ is a blocking sequence then arguing as before we obtain

$$Q(b) \leq Q(b_{2q+1}) + \sum_{k=1}^q m_{\beta(b_{2k-1}), \beta(b_{2k})} \leq m_{\beta(b_{2q+1}), l} - 1 + \sum_{k=1}^q m_{\beta(b_{2k-1}), \beta(b_{2k})} \leq m_{ij} - 1,$$

with equality implying $1 = P(b) \geq P(b_{2q+1}) + d - 1$, where d is the number of distinct boxes appearing in the sequence $b_0, b_1, \dots, b_{2q+1}$. It follows that $d = 1$, that $q = 0$, that $b_0 = b_1$, and that

$$d_{\beta(b_1)} - d_l + r\text{ct}(b_1)c_0 = m_{il}.$$

But now we have $m_{ij} \leq m_{il}$ since $(P, Q) \in \Gamma_c$, and the opposite inequality follows from the definition of blocking sequence. Thus $m_{ij} = m_{il}$, whence $l = j$ and we are done. \square

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